1. MATHEMATICAL INDUCTION

**EXAMPLE 1:** Prove that

\[ 1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2} \]  \hspace{1cm} (1.1)

for any integer \( n \geq 1 \).

**Proof:**

**STEP 1:** For \( n=1 \) (1.1) is true, since

\[ 1 = \frac{1(1+1)}{2}. \]

**STEP 2:** Suppose (1.1) is true for some \( n = k \geq 1 \), that is

\[ 1 + 2 + 3 + \ldots + k = \frac{k(k + 1)}{2}. \]

**STEP 3:** Prove that (1.1) is true for \( n = k + 1 \), that is

\[ 1 + 2 + 3 + \ldots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}. \]

We have:

\[
1 + 2 + 3 + \ldots + k + (k + 1) \overset{\text{ST.2}}{=} \frac{k(k + 1)}{2} + (k + 1) = (k + 1) \left( \frac{k}{2} + 1 \right) = \frac{(k + 1)(k + 2)}{2}. \]

**EXAMPLE 2:** Prove that

\[ 1 + 3 + 5 + \ldots + (2n - 1) = n^2 \]  \hspace{1cm} (1.2)

for any integer \( n \geq 1 \).

**Proof:**

**STEP 1:** For \( n=1 \) (1.2) is true, since \( 1 = 1^2 \).

**STEP 2:** Suppose (1.2) is true for some \( n = k \geq 1 \), that is

\[ 1 + 3 + 5 + \ldots + (2k - 1) = k^2. \]

**STEP 3:** Prove that (1.2) is true for \( n = k + 1 \), that is

\[ 1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) \overset{?}{=} (k + 1)^2. \]

We have: \[ 1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) \overset{\text{ST.2}}{=} k^2 + (2k + 1) = (k + 1)^2. \]
EXAMPLE 3: Prove that
\[ n! \leq n^n \] (1.3)
for any integer \( n \geq 1 \).

**Proof:**

STEP 1: For \( n=1 \) (1.3) is true, since \( 1! = 1^1 \).

STEP 2: Suppose (1.3) is true for some \( n = k \geq 1 \), that is \( k! \leq k^k \).

STEP 3: Prove that (1.3) is true for \( n = k + 1 \), that is \((k + 1)! \leq (k + 1)^{k+1}\). We have
\[(k + 1)! = k! \cdot (k + 1) \leq k^k \cdot (k + 1) < (k + 1)^k \cdot (k + 1) = (k + 1)^{k+1}. \]

EXAMPLE 4: Prove that
\[ 8 \mid 3^{2n} - 1 \] (1.4)
for any integer \( n \geq 0 \).

**Proof:**

STEP 1: For \( n=0 \) (1.4) is true, since \( 8 \mid 3^0 - 1 \).

STEP 2: Suppose (1.4) is true for some \( n = k \geq 0 \), that is \( 8 \mid 3^{2k} - 1 \).

STEP 3: Prove that (1.4) is true for \( n = k + 1 \), that is \( 8 \mid 3^{2(k+1)} - 1 \). We have
\[ 3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k} \cdot 9 - 1 = 3^{2k}(8 + 1) - 1 = \underbrace{3^{2k} \cdot 8}_{\text{div. by 8}} + \underbrace{3^{2k} - 1}_{\text{St. 2}}. \]

EXAMPLE 5: Prove that
\[ 7 \mid n^7 - n \] (1.5)
for any integer \( n \geq 1 \).

**Proof:**

STEP 1: For \( n=1 \) (1.5) is true, since \( 7 \mid 1^7 - 1 \).

STEP 2: Suppose (1.5) is true for some \( n = k \geq 1 \), that is \( 7 \mid k^7 - k \).

STEP 3: Prove that (1.5) is true for \( n = k + 1 \), that is \( 7 \mid (k + 1)^7 - (k + 1) \). We have
\[ (k + 1)^7 - (k + 1) = k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1 - k - 1 = \underbrace{k^7 - k}_{\text{St. 2}} + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k. \]
[div. by 7]